Module 2: Portfolio Theory

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Summary: This module introduces the fundamental principles of Modern Portfolio Theory, originally developed by Harry Markowitz. Students will learn how to construct portfolios that balance risk and return through asset diversification. The module explains how to calculate the expected return and variance of a portfolio, and how the correlation between assets affects overall portfolio risk. Using optimization techniques, the module derives the Efficient Frontier, which describes the optimal combinations of assets for different levels of expected return. It also presents the Minimum Variance Portfolio, which achieves the lowest possible volatility. The module then explores portfolio efficiency through the Sharpe Ratio, showing how to identify the Tangency Portfolio, which maximizes return per unit of risk. Finally, it introduces the Capital Allocation Line (CAL), which illustrates optimal combinations of risky assets and a risk-free asset, enabling leverage and risk-adjusted investment strategies. This approach provides students with a solid foundation to understand and apply quantitative methods in portfolio construction.

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1 Markowitz's Optimization

Even though Modern Portfolio Theory remains one of the most imperative methodologies in the portfolio management business, Markowitz's Theory stands as the foundational pillar that theoretically sustains all modern approaches to understanding stock markets and the economic mechanisms that influence our world today. Originally developed in the 1950s, Markowitz's insights into diversification and risk optimization transformed investment strategy from simple asset selection to a structured process aimed at maximizing returns for a given level of risk. This approach has since become essential to portfolio construction, influencing not only academic models but also practical decision-making in financial institutions globally.

1.1 Portfolio's Returns and Variance

As read in Markowitz, 1952, the expected return of a given portfolio P is a weighted average of the returns of all the assets it comprises, with weights ω_i determined according to the portfolio manager's chosen criteria. Evidently, the sum of the weights must equal one ($\sum \omega_i = 1$). The expected returns are also a weighted average, where the weights are the probabilities of specific values occurring.

$$\mu_P = \sum_{i=1}^{n} \omega_i \mu_i \tag{1}$$

Then it is important to also understand how to measure the risk of a portfolio:

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \gamma_i j \tag{2}$$

We can note that the variance of a portfolio's returns depends on the covariances ($\gamma_i j$) between each pair of assets within the portfolio. This leads to one of the main conclusions discussed in Portfolio Theory: diversification can reduce a portfolio's volatility (risk) if the assets are uncorrelated. This effect becomes clear when we separate the previous expression into two parts:

$$\sigma_P^2 = \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \omega_i \omega_j \gamma_i j \tag{3}$$

Evidently, the variance of the portfolio will be greater if the covariance among the assets is positive and nonzero. When assets are uncorrelated, the portfolio's variance equals the sum of the variances of each asset, since ($\gamma_i j = 0$):

$$\sigma_P^2 = \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 \tag{4}$$

Given the formula for Pearson Correlation (see Module 1 for more information), we can conclude that $\gamma_{ij} = \sigma_i \sigma_j$, if the stocks are perfectly correlated ($\rho_{ij} = 1$). If the stocks are not perfectly correlated ($\rho_{ij} < 1$) then the covariance would be $\gamma_{ij} = \rho_{ij} \sigma_i \sigma_j < \sigma_i \sigma_j$ (Elton et al., 2014). Thus, we can conclude that the maximum value of the portfolio's variance would be represented by the previously shown expression.The opposite occurs when the stocks are perfectly negatively correlated; then, the variance will fall within the following range:

$$\sigma_P^2 \in \left[\sum_{i=1}^{n} \omega_i^2 \sigma_i^2 - \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \omega_i \omega_j \sigma_i \sigma_j , \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \omega_i \omega_j \sigma_i \sigma_j \right] \tag{5}$$

1.2 Diversification

The definition of the correlation coefficient leads us to a question that is interesting to say the least. What would you rather have in a diversified portfolio? On the one hand, you have the option of including financial assets that have zero correlations with each other and whose returns are independent of each other. On the other hand, you have the option of including highly correlated financial assets, which means that the losses of one stock could be mitigated by the gains of others. In which of the two cases do you better mitigate market risk?

To do this, as read in Francis and Kim, 2013, we must understand how we obtain the return of a portfolio of diversified investments. This is given by the weighted average of the returns of the stocks or financial assets that make up the portfolio. Weights (referred to as ω) can be defined in different ways, however, we will assume that our portfolio is an 'equal weighted' portfolio, implying that $\omega_1 = \omega_2 = ... = \omega_n$. If we also assume that there are only two assets in our portfolio, the return of the portfolio will be:

$$\mu_P = \omega\left(\mu_1 + \mu_2\right) \tag{6}$$

Now, we can derive the change in portfolio returns as the (weighted) sum of the changes in stock returns:

$$\Delta\mu_P = \omega\left(\Delta\mu_1 + \Delta\mu_2\right) \tag{7}$$

By setting the previous equation equal to zero, we obtain the following equivalence:

$$\Delta\mu_1 = -\Delta\mu_2 \tag{8}$$

What does this suggest? Are we finding that the way to maximize returns from a portfolio is to choose from stocks that are oppositely correlated with each other? It would seem that the answer is mathematically obvious, however, in portfolios they are not always of homogeneous weights or of only two assets. However, it would not be absurd to assume that the best way to manage risk is to choose stocks or financial assets that offset their losses against each other, such that:

$$\Delta\mu_1 = -\frac{1}{\omega_1}\sum_{i=2}^{n}\omega_i\Delta\mu_i \tag{9}$$

This would imply that, in order to maximize returns, we should make sure that the losses generated by a single asset are offset by the sum of the returns of all the other stocks in the portfolio, only in the opposite direction. This would help us to conclude that, at the very least, it is convenient to consider assets that are negatively correlated with each other. Just remember that, if the correlation between two assets is -1, this does NOT imply that the magnitudes of the changes are equal, it only tells us that they will always go in the opposite direction.

1.3 Mean-Variance Optimization

The purpose of the optimization problem proposed by Markowitz, 1952, is to minimize the variance of a portfolio subject to the following constraints: (1) the return of the portfolio is defined as a weighted average of the returns of the assets composing the portfolio, and (2) the sum of the weights equals one. We can then propose a utility-based Lagrangian function (in matrix form) to optimize, as read in Petters and Dong, 2016.

$$\mathcal{L}\left(\omega,\lambda_1,\lambda_2\right) = \omega^\top\Sigma\omega - \lambda_1\left(\omega^\top\mu - \mu_P\right) - \lambda_2\left(\omega^\top\iota - 1\right) \tag{10}$$

Here, ω is the weights vector for each stock in the portfolio, with dimensions $n\times 1$. The vector μ contains the expected returns of the assets and also has dimensions $n\times 1$, while μ_P is a scalar representing the desired level of returns for the portfolio. Finally, Σ is the covariance matrix (with $n\times n$ dimensions) of the stocks, which includes the variances of the assets (on its diagonal) and the covariances between them. The vector ι is simply an $n \times 1$ vector of one. All the parts of the equation are products of the matrix that produce scalars. Then we derive the equation to obtain the first order conditions:

$$\frac{\partial \mathcal{L}(\omega, \lambda_1, \lambda_2)}{\partial \omega} = 2\Sigma\omega - \lambda_1 \mu - \lambda_2 \iota = 0 \tag{11}$$

Now let us see the other conditions:

$$\frac{\partial \mathcal{L}(\omega, \lambda_1, \lambda_2)}{\partial \lambda_1} = \omega^\top \mu - \mu_P = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}(\omega, \lambda_1, \lambda_2)}{\partial \lambda_2} = \omega^\top \iota - 1 = 0 \tag{13}$$

If we isolate all the first order conditions, we can obtain the next system of equations to solve. The expression $\Sigma\omega$ represents a vector that contains the first n First Order Conditions, just as the expression that we analyzed in the previous section; it can be also expressed as the covariance of each asset i with the rest of stocks in the portfolio. Now we can create a new system of equations with a matrix form (Petters and Dong, 2016). Note that by solving the matrix multiplication we are obtaining the First Order Conditions from above:

$$\begin{bmatrix} \Sigma & -\frac{\mu}{2} & -\frac{\iota}{2} \\ \mu^\top & 0 & 0 \\ \iota^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_P \\ 1 \end{bmatrix} \tag{14}$$

By isolating the vector of weights in the first expression we saw previously, we can obtain the form of the optimal weights, which is a function that depends negatively on the covariance matrix and positively on the Lagrange multipliers:

$$\omega = \frac{\lambda_1}{2}\Sigma^{-1}\mu + \frac{\lambda_2}{2}\Sigma^{-1}\iota \tag{15}$$

Replacing this new expression in the second First Order Condition (equation 12), which is also a restriction for the optimization problem ($\omega^\top \mu = \mu_P$):

$$\left(\frac{\lambda_1}{2}\Sigma^{-1}\mu + \frac{\lambda_2}{2}\Sigma^{-1}\iota\right)^\top \mu = \mu_P \tag{16}$$

Then, we are going to rename some terms to simplify mathematics, let us call $a = \frac{\lambda_1}{2}$ and $b = \frac{\lambda_2}{2}$. So, then we can expand the parenthesis and obtain:

$$a\left(\mu^\top \Sigma^{-1}\mu\right) + b\left(\iota^\top \Sigma^{-1}\mu\right) = \mu_P \tag{17}$$

Then doing the same for equation 13:

$$a\left(\mu^\top \Sigma^{-1}\iota\right) + b\left(\iota^\top \Sigma^{-1}\iota\right) = 1 \tag{18}$$

We can redefine some terms:

$$A = \mu^\top \Sigma^{-1}\mu \tag{19}$$

$$B = \mu^\top \Sigma^{-1}\iota \tag{20}$$

$$C = \iota^\top \Sigma^{-1}\iota \tag{21}$$

Note that A, B, and C are all scalar values, not vectors nor arrays (this can be verified by examining the dimensions of the arrays that make them up). Consequently, we can rewrite the expressions for the First Order Conditions as a simple system of linear equations, as follows:

$$aA + bB = \mu_P \tag{22}$$
$$aB + bC = 1 \tag{23}$$

By solving this system of equations we can arrive to the optimal values of the Lagrange Multipliers $a = \frac{\lambda_1}{2}$ and $b = \frac{\lambda_2}{2}$:

$$\Lambda = \{\lambda_1 = 2\frac{\mu_p C - B}{AC - B^2}, \lambda_2 = 2\frac{A - \mu_p B}{AC - B^2}\} \tag{24}$$

By substituting these equations into the expression for the optimal weights, we can derive the final formula to calculate them:

$$\omega^* = \frac{\mu_p C - B}{AC - B^2}\Sigma^{-1}\mu + \frac{A - \mu_p B}{AC - B^2}\Sigma^{-1}\iota \tag{25}$$

1.4 Efficient Frontier Derivation

Continuing with the derivation (Petters and Dong, 2016), and after we found the optimal weights, we can derive the minimum variance equation for a given desired level of returns for our portfolio. To do so, we must substitute equation 25 into the original equation for the portfolio variance $\left(\sigma_P^2 = \omega^\top \Sigma \omega\right)$:

$$\sigma_P^2 = \left(\frac{\mu_p C - B}{AC - B^2}\Sigma^{-1}\mu + \frac{A - \mu_p B}{AC - B^2}\Sigma^{-1}\iota\right)^\top \Sigma \left(\frac{\mu_p C - B}{AC - B^2}\Sigma^{-1}\mu + \frac{A - \mu_p B}{AC - B^2}\Sigma^{-1}\iota\right) \tag{26}$$

Distribute for Σ into the second parentheses so that the covariance matrix is canceled. Then by multiplying both parentheses we will arrive in an expanded form:

$$\sigma_P^2 = \left(\frac{\mu_p C - B}{AC - B^2}\Sigma^{-1}\mu + \frac{A - \mu_p B}{AC - B^2}\Sigma^{-1}\iota\right)^\top \left(\frac{\mu_p C - B}{AC - B^2}\mu + \frac{A - \mu_p B}{AC - B^2}\iota\right) \tag{27}$$

Expanding the equation by multiplying the components of the parentheses:

$$\sigma_P^2 = \left(\frac{\mu_p C - B}{AC - B^2}\right)^2 \mu^\top \Sigma^{-1}\mu + \left(\frac{A - \mu_p B}{AC - B^2}\right)^2 \iota^\top \Sigma^{-1}\iota + 2\frac{(\mu_P C - B)(A - \mu_P B)}{(AC - B^2)^2}\mu^\top \Sigma^{-1}\iota \tag{28}$$

Note all these numbers are the same scalars we saw on previous equations. Recalling the forms of A, B, and C:

$$\sigma_P^2 = \left(\frac{\mu_p C - B}{AC - B^2}\right)^2 A + \left(\frac{A - \mu_p B}{AC - B^2}\right)^2 C + 2\frac{(\mu_P C - B)(A - \mu_P B)}{(AC - B^2)^2} B \tag{29}$$

Now we have to expand the numerators of the right side of the equation so we can better handle them with algebra. Next, we show you these expanded versions:

$$(\mu_p C - B)^2 = \mu_P^2 C^2 + B^2 - 2\mu_P BC \tag{30}$$
$$(A - \mu_p B)^2 = A^2 + \mu_P^2 B^2 - 2\mu_P AB \tag{31}$$
$$(\mu_p C - B)(A - \mu_p B) = \mu_P AC - \mu_P^2 BC - AB + \mu_P B^2 \tag{32}$$

After substituting these new forms on the previous equation:

$$\sigma_P^2 = \frac{A\left(\mu_P^2 C^2 + B^2 - 2\mu_P BC\right) + C\left(A^2 + \mu_P^2 B^2 - 2\mu_P AB\right) + 2B\left(\mu_P AC - \mu_P^2 BC - AB + \mu_P B^2\right)}{\left(AC - B^2\right)^2} \tag{33}$$

And then distributing A, $2B$, and C on the new parentheses:

$$\sigma_P^2 = \frac{\mu_P^2 AC^2 + AB^2 - 2\mu_P ABC + A^2C + \mu_P^2 B^2 C - 2\mu_P ABC + 2\mu_P ABC - 2\mu_P^2 B^2 C - 2AB^2 + 2\mu_P B^3}{\left(AC - B^2\right)^2} \tag{34}$$

Now we are going to cancel some terms like $2\mu_P ABC$ and $-2\mu_P ABC$. And then, we can regroup some terms that share common characteristics:

$$\sigma_P^2 = \frac{\left(\mu_P^2 AC^2 - \mu_P^2 B^2 C\right) + \left(2\mu_P B^3 - 2\mu_P ABC\right) + \left(A^2C - AB^2\right)}{\left(AC - B^2\right)^2} \tag{35}$$

Then find the common factors of the new groups, such as $\mu_P^2 C$ on the first term, $-2\mu_P B$ on the second one, and A on the third one:

$$\sigma_P^2 = \frac{\mu_P^2 C\left(AC - B^2\right) - 2\mu_P B\left(-B^2 + AC\right) + A\left(AC - B^2\right)}{\left(AC - B^2\right)^2} \tag{36}$$

Finally, we can cancel the $\left(AC - B^2\right)$ term since all the groups share it. Then we will find the minimum variance for a portfolio given the desired level of returns:

$$\sigma_P^2 = \frac{\mu_P^2 C - 2\mu_P B + A}{(AC - B^2)} \tag{37}$$

Let us redefine some terms to obtain a more concise form. Hence, we are going to define $\pi_2 = \frac{C}{(AC-B^2)}$, $\pi_1 = \frac{2B}{(AC-B^2)}$, and $\pi_0 = \frac{A}{(AC-B^2)}$. Now, we have derived a function $\sigma_P^2(\mu_P)$ in the form of a quadratic equation, specified as:

$$\sigma_P^2 = \pi_0 - \pi_1\mu_P + \pi_2\mu_P^2 \tag{38}$$

This equation expresses the minimum variance for each portfolio as a parabolic function, just as Harry Markowitz proposed (Markowitz, 1952), dependent on the expected returns of those portfolios.

1.5 The Minimum Variance Portfolio (MVP)

In the Efficient Frontier itself, some portfolios minimize variance without maximizing returns, while others strike an optimal balance. The vertex of the frontier, representing the portfolio with the minimum variance, marks a unique point: at this level of variance, only one portfolio exists on the curve. Beyond this vertex, any given level of variance typically corresponds to two distinct portfolios offering different returns. Naturally, the portfolio that maximizes returns will always be preferred (Merton, 1972). The vertex can be found where the returns are:

$$\mu_{MVP} = \frac{\pi_1}{2\pi_2} \tag{39}$$

This form can be derived from the parabolic equation of the Efficient Frontier. The inflection point of the parabola can be determined by substituting the previous expression into the equation. The goal is to identify the variance value that corresponds to the absolute minimum variance.

$$\sigma_P^2\left(\frac{\pi_1}{2\pi_2}\right) = \pi_0 - \pi_1\left(\frac{\pi_1}{2\pi_2}\right) + \pi_2\left(\frac{\pi_1}{2\pi_2}\right)^2 \tag{40}$$

We can expand and simplify the values within the parentheses on the right side of the equation. Afterward, we can simplify the fractions to obtain a new expression for the minimum variance, as shown below:

$$\sigma_{MVP}^2 = \pi_0 - \frac{\pi_1^2}{4\pi_2} \tag{41}$$

Given this form of the variances, we will select the best portfolios that lie on the Efficient Frontier and maximize returns for a given level of volatility. Let P represent the set of these optimal efficient portfolios that maximize returns and minimize volatility.

$$P = \left\{\left(\sigma_P^2, \mu_P\right) \;:\; \sigma_P^2(\mu_P) = \pi_0 - \pi_1\mu_P + \pi_2\mu_P^2, \quad \forall\, \mu_P \geq \mu_{MVP} = \frac{\pi_1}{2\pi_2}\right\} \tag{42}$$

2 The Most Efficient Portfolio

To assess a portfolio's efficiency, one commonly used tool in portfolio management is the Sharpe Ratio. As mentioned in Module 1, the Sharpe Ratio evaluates a portfolio's returns adjusted for risk, providing a measure of how well the portfolio performs relative to its volatility (Sharpe, 1966). Specifically, it is defined as the excess return of a portfolio over the risk-free rate (r_f), adjusted for the portfolio's volatility. Higher values of the Sharpe Ratio indicate a more favorable risk-return profile, helping investors identify more efficient portfolios.

2.1 Sharpe Ratio Maximization

If you recall the shape of the Efficient Frontier, you'll remember that all portfolios on its upper part will have the highest Sharpe Ratios for their respective levels of risk. In other words, every portfolio below the frontier has a superior counterpart in terms of returns, and thus in terms of efficiency. Among all the efficient portfolios, however, there is one that stands out as the most efficient of them all. To find it, we can solve an optimization problem (as you can read in Lennartsson and Ekman, 2024) that involves the maximization of the Sharpe Ratio subject to the parabolic equation of the Efficient Frontier.

$$S = \frac{\omega^\top\mu - r_f}{\sqrt{\omega^\top\Sigma\omega}} \tag{43}$$

Note that we are now combining the two fundamental equations of Markowitz's optimization into the equation of the Sharpe Ratio (the returns and the variance of a portfolio), so the only restriction will be that the sum of the weights equals one, ensuring the portfolio is fully invested. We can then propose the Lagrangian equation as follows:

$$\mathcal{L}\left(\omega, \theta_1\right) = \frac{\omega^\top\mu - r_f\iota}{\omega^\top\Sigma\omega} - \theta_1\left(\omega^\top\iota - 1\right) \tag{44}$$

However, solving the optimization problem proposed in matrix form can be particularly complicated, so we can propose an alternative formulation in which we maximize the active (excess) returns subject to the portfolio variance being equal to one. The reader can find these alternative optimization problems in Zivot, 2021

$$\mathcal{L}(\omega, \theta_1) = \omega^\top (\mu - r_f \iota) - \theta_1 \left(\omega^\top \Sigma \omega - 1\right) \tag{45}$$

You can prove that if the sum of the weights equals one, the numerator of Equation 44 and the objective function in Equation 45 are essentially the same. In this way, we are implicitly including the restriction of a fully invested portfolio. The reason we are "forcing" the variance to be one is that, mathematically, this approach allows us to transform the optimization problem by normalizing the scales; this adjustment will be addressed later.

$$\omega = \frac{1}{2\theta_1} \Sigma^{-1} (\mu - r_f \iota) \tag{46}$$

The first step, as always, is to find the first partial derivatives. After doing so, we can derive the optimal form for the portfolio weights. Equation 46 shows the result of rearranging the first partial derivative of the Lagrangian function (Equation 45) with respect to the weights. We can then take this result and substitute it into the constraint function:

$$\left[\frac{1}{2\theta_1} \Sigma^{-1} (\mu - r_f \iota)\right]^\top \Sigma \left[\frac{1}{2\theta_1} \Sigma^{-1} (\mu - r_f \iota)\right] = 1 \tag{47}$$

We can rearrange some terms and cancel others to obtain the following expression:

$$\left(\frac{1}{2\theta_1}\right)^2 (\mu - r_f \iota)^\top \Sigma^{-1} (\mu - r_f \iota) = 1 \tag{48}$$

After solving this equation for the Lagrange multiplier (θ_1), we obtain its optimal value:

$$\theta_1 = \frac{1}{2} \sqrt{(\mu - r_f \iota)^\top \Sigma^{-1} (\mu - r_f \iota)} \tag{49}$$

Substituting the Lagrange multiplier into Equation 46 yields the optimal form of the weights that maximizes the Sharpe Ratio:

$$\omega = \frac{\Sigma^{-1} (\mu - r_f \iota)}{\sqrt{(\mu - r_f \iota)^\top \Sigma^{-1} (\mu - r_f \iota)}} \tag{50}$$

This is not the final result, because in our optimization setup we rescaled the variance to simplify the algebra. We must now normalize the weights so that they sum to one—that is, divide the weight vector by its own sum, equivalently by its inner product with the vector of ones ι:

$$\frac{\omega}{\iota^\top \omega} = \frac{\frac{\Sigma^{-1} (\mu - r_f \iota)}{\sqrt{(\mu - r_f \iota)^\top \Sigma^{-1} (\mu - r_f \iota)}}}{\iota^\top \frac{\Sigma^{-1} (\mu - r_f \iota)}{\sqrt{(\mu - r_f \iota)^\top \Sigma^{-1} (\mu - r_f \iota)}}} \tag{51}$$

If Equation 51 looks intimidating, there is no need to worry, we can simply cancel out the square root terms in both the numerator and the denominator, since they are identical. This gives us a simpler expression. The weights for the maximum Sharpe ratio portfolio are then:

$$\omega_T = \frac{\Sigma^{-1} (\mu - r_f \iota)}{\iota^\top \Sigma^{-1} (\mu - r_f \iota)} \tag{52}$$

To calculate the portfolio returns, we simply multiply the maximizing weights by the return vector μ:

$$\mu_T = \omega_T^\top \mu = \left(\frac{\Sigma^{-1}(\mu - r_f \iota)}{\iota^\top \Sigma^{-1}(\mu - r_f \iota)} \right)^\top \mu \tag{53}$$

We can now distribute the vector of returns and obtain a new expression:

$$\mu_T = \frac{\mu^\top \Sigma^{-1} \mu - r_f \mu^\top \Sigma^{-1} \iota}{\mu^\top \Sigma^{-1} \iota - r_f \iota^\top \Sigma^{-1} \iota} \tag{54}$$

Recalling Equations 19, 20 and 21, we can transform the notation of the previous expression:

$$\mu_T = \frac{A - r_f B}{B - r_f C} \tag{55}$$

Finally, we can multiply the previous expression by $\frac{2}{AC-B^2}$ to obtain the next equation, which uses the coefficients of the parabolic function of the efficient frontier:

$$\mu_T = \frac{2\pi_0 - \pi_1 r_f}{\pi_1 - 2\pi_2 r_f} \tag{56}$$

You can demonstrate that, since the Efficient Frontier has no real solution (discriminant less than zero), we can conclude that the tangency portfolio is superior to the minimum variance portfolio (MVP).

2.2 The Tangency Portfolio

For the derivation of the variance of the tangency portfolio, we have to replace the tangency returns form on the variance parabolic equation:

$$\sigma_P^2 = \pi_0 - \pi_1 \left(\frac{2\pi_0 - \pi_1 r_f}{\pi_1 - 2\pi_2 r_f} \right) + \pi_2 \left(\frac{2\pi_0 - \pi_1 r_f}{\pi_1 - 2\pi_2 r_f} \right)^2 \tag{57}$$

Then we can solve both the parentheses, distributing the constants π_1 and π_2, and solving the squares of the binomials of the third component:

$$\sigma_P^2 = \pi_0 - \frac{2\pi_0\pi_1 - \pi_1^2 r_f}{\pi_1 - 2\pi_2 r_f} + \frac{\pi_!^2 \pi_2 r_f^2 + 4\pi_0^2\pi_2 - 4\pi_0\pi_1\pi_2}{\left(\pi_1 - 2\pi_2 r_f\right)^2} \tag{58}$$

To add fractions all the components must have the same denominator $(\pi_1 - 2\pi_2 r_f)^2$, so we are multiplying the second term by $(\pi_1 - 2\pi_2 r_f)$ in both the numerator and denominator, which is the same as multiplying by one, so the equality is unchanged:

$$\sigma_P^2 = \pi_0 - \frac{(\pi_1 - 2\pi_2 r_f)\left(2\pi_0\pi_1 - \pi_1^2 r_f\right)}{\left(\pi_1 - 2\pi_2 r_f\right)^2} + \frac{\pi_!^2 \pi_2 r_f^2 + 4\pi_0^2\pi_2 - 4\pi_0\pi_1\pi_2}{\left(\pi_1 - 2\pi_2 r_f\right)^2} \tag{59}$$

After expanding the parentheses and rearranging terms, we arrive at a more simplified form. Note that certain terms can be canceled and others combined, allowing us to obtain a more concise expression:

$$\sigma_P^2 = \pi_0 + \frac{\pi_1^2 \pi_2 r_f^2 + 4\pi_0^2\pi_2 - 4\pi_0\pi_1\pi_2 r_f - 2\pi_1^2\pi_2 r_f^2 + 4\pi_0\pi_1\pi_2 r_f + \pi_1^3 r_f - 2\pi_0\pi_1^2}{\left(\pi_1 - 2\pi_2 r_f\right)^2} \tag{60}$$

Then, after canceling some equal terms, we multiply the first term by the denominator of the fraction, to combine all the terms:

$$\sigma_P^2 = \pi_0 \frac{(\pi_1 - 2\pi_2 r_f)^2}{(\pi_1 - 2\pi_2 r_f)^2} + \frac{4\pi_0^2\pi_2 - \pi_1^2\pi_2 r_f^2 + \pi_1^3 r_f - 2\pi_0\pi_1^2}{(\pi_1 - 2\pi_2 r_f)^2} \tag{61}$$

By solving the squared binomial and distributing π_0, we obtain the following equation:

$$\sigma_P^2 = \frac{4\pi_0\pi_2^2 r_f^2 + \pi_0\pi_1^2 - 4\pi_0\pi_1\pi_2 r_f + 4\pi_0^2\pi_2 - \pi_1^2\pi_2 r_f^2 + \pi_1^3 r_f - 2\pi_0\pi_1^2}{(\pi_1 - 2\pi_2 r_f)^2} \tag{62}$$

Now, let us regroup those terms that share common elements like r_f^2 or r_f. Then we can take common factors so the equation will have the next form:

$$\sigma_P^2 = \frac{\pi_2 r_f^2(4\pi_0\pi_2 - \pi_1^2) - \pi_1 r_f(4\pi_0\pi_2 - \pi_1^2) + \pi_0(4\pi_0\pi_2 - \pi_1^2)}{(\pi_1 - 2\pi_2 r_f)^2} \tag{63}$$

And finally, by taking the last common factor, we will obtain the variance of the portfolio that maximizes the Sharpe Ratio:

$$\sigma_T^2 = \frac{(4\pi_0\pi_2 - \pi_1^2)(\pi_2 r_f^2 - \pi_1 r_f + \pi_0)}{(\pi_1 - 2\pi_2 r_f)^2} \tag{64}$$

Note that the second parenthesis in the numerator represents the variance of a portfolio whose returns are equal to those of the risk-free asset.

$$\sigma_T^2 = \frac{(4\pi_0\pi_2 - \pi_1^2)}{(\pi_1 - 2\pi_2 r_f)^2}\sigma_P^2(r_f) \tag{65}$$

This portfolio, known as the Tangent Portfolio, is the most efficient portfolio in terms of the Sharpe Ratio. It maximizes the excess return over the risk-free rate for a given level of risk, achieving the highest return per unit of volatility. In other words, it represents the optimal balance between risk and return on the Efficient Frontier. A risk-averse investor would be highly satisfied with selecting this portfolio, as it provides the highest possible compensation for taking on additional risk.

$$T : \left[\sqrt{\frac{(4\pi_0\pi_2 - \pi_1^2)}{(\pi_1 - 2\pi_2 r_f)^2}\sigma_P^2(r_f)}, \frac{2\pi_0 - \pi_1 r_f}{\pi_1 - 2\pi_2 r_f}\right] \tag{66}$$

2.3 Capital Allocation Line

Now, consider a different portfolio that is not composed of risky assets (those stocks with a variance greater than zero), but consists only of the non-risky asset, which pays the risk-free rate r_f as returns. Of course, this portfolio will have a variance of zero, and we will locate it at point $F : [0, r_f]$. We can then create a line that crosses through both points T and F (as shown in Petters and Dong, 2016). Remember the equation of a slope:

$$m = \frac{\mu_T - r_f}{\sigma_T - 0} \tag{67}$$

Note that the slope of this linear equation is indeed the Sharpe Ratio of the Tangency Portfolio. By intuition, we know that all portfolios on this line will have the same Sharpe Ratio as the tangent point. Now, we can use the equation for the midpoint of a line, $y - y_1 = m(x - x_1)$, with the coordinates of points T and F:

$$\mu_P = r_f + \frac{\mu_T - r_f}{\sigma_T}\sigma_P \tag{68}$$

Then, by simplifying the equation, we arrive at the well-known Capital Allocation Line (CAL). This line illustrates all possible combinations between a risky portfolio (composed of volatile assets like stocks) and a risk-free asset (such as a treasury bond). The CAL also shows how an investor can allocate wealth between risky and non-risky assets. An investor may choose to sacrifice returns for lower volatility by lending funds and investing in risk-free assets, or alternatively, take on more risk by borrowing funds to achieve higher returns (leverage).

$$CAL: \ \mu_P = r_f + S_T(r_f, \pi_0, \pi_1, \pi_2)\sigma_P \tag{69}$$

Note that the shape of the Capital Allocation Line depends on market information, since the coefficients of the Efficient Frontier — and hence the Sharpe Ratio of the Tangency Portfolio — depend on the returns and covariances of the assets included in the portfolio.

2.4 CAL Portfolio's Weights

Since the Capital Allocation Line gives us the option of lending or borrowing to adjust returns for a given level of risk, we can conclude that every portfolio on the CAL will have two components: (1) a subportfolio of risky assets and (2) the risk-free asset.

$$\mu_P = \sum_{i=1}^{n} \omega_i \mu_i + \omega_f r_f \tag{70}$$

Since the weights of the portfolio must sum 1:

$$\mu_P = \sum_{i=1}^{n} \omega_i \mu_i + \left(1 - \sum_{i=1}^{n} \omega_i\right) r_f \tag{71}$$

Then our wealth will be distributed between a risky portfolio and a risk-free asset. Suppose that risky portfolio is the Tangency Portfolio:

$$\mu_P = \omega_T \mu_T + (1 - \omega_T)\, r_f \tag{72}$$

Then, the fraction of our wealth that we are going to allocate in the risky portfolio will be defined as follows:

$$\omega_R = \frac{\mu_P - r_f}{\mu_T - r_f} \tag{73}$$

Since the tangency portfolio is equivalent, in terms of the Sharpe ratio, to all portfolios on the Capital Allocation Line, we can use Equation 73 to determine the level of leverage required to achieve extraordinary returns. This concept of lending or borrowing is elegantly described in Tobin, 1958, where he relates the inclusion of the risk-free asset in our portfolios to our risk aversion.

- If $\mu_P > \mu_T$, then $\omega_T > 1$, which means that, if our desired returns are larger than the returns of the tangency portfolio, we will invest more than our available wealth in risky assets. How? By borrowing money (at the risk-free rate). This is called leverage. This means that we are selecting a portfolio superior to the Tangency Portfolio.

- If $\mu_P = \mu_T$, then $\omega_T = 1$, which means that, if our desired returns are equal than the returns of the tangency portfolio, we invest exactly our available wealth in a portfolio of risky assets, in this case the Tangency Portfolio. This means that we are not including the risk-free asset in our portfolio.
- If $\mu_P < \mu_T$, then $\omega_T < 1$, which means that, if our desired returns are smaller than the returns of the tangency portfolio, we will invest just a fraction of our available wealth in risky assets. The rest will be located on the risk-free rate. This might be understood as lending money to others.

In this case, to find the combination of weights that will build our desired portfolio in the capital allocation line, we only need to apply the adjustment factor ω_R, to the weights of the tangency portfolio.

$$\omega_{CAL} = \frac{\mu_P - r_f}{\mu_T - r_f}\omega_T \tag{74}$$

Remember ω_T is the vector of weights that build the tangency portfolio, that we calculated in Equation 66.

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