Module 6: Performance and Risk Attribution

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Summary: This module provides a comprehensive exploration of portfolio performance and risk attribution methodologies, focusing on three influential frameworks: the Brinson-Fachler model, the Brinson-Hood-Beebower (BHB) model, and the Menchero risk attribution approach. We begin by analytically deriving the allocation, selection, and interaction effects proposed by Brinson and Fachler, clarifying their interpretation at the sector level. We then transition to the BHB model, which reframes the original decomposition using a more intuitive four-portfolio structure, emphasizing investment policy, market timing, and security selection. Finally, we examine Menchero's x-sigma-rho decomposition, which attributes portfolio risk based on asset exposures, volatilities, and correlations. The module also introduces the concepts of marginal and percentage contributions to risk, and presents a correlation drilldown technique to better understand the interdependencies among assets. Through this structured progression, we equip readers with a solid foundation in both return- and risk-based attribution techniques.

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1 Brinson-Fachler Model

In 1985, Gary P. Brinson and Nimrod Fachler developed one of the most influential methodologies for understanding portfolio performance attribution. They proposed a framework that captures the effects of investors' active strategies relative to a benchmark (or the market). Brinson and Fachler recognized that, in order to generate excess returns, investors cannot influence the market to favor the assets in their portfolios. In a competitive market, no investor has the power to 'inflate' stock prices to achieve better returns. What they can do, however, is construct different combinations of assets and adjust their weights within the portfolio to generate superior performance.

This is what we refer to as an **active strategy**. For example, well-informed investors may assign greater weights to stocks that outperform the benchmark (e.g., NVDA versus the S&P 500), and reduce the weights of those with weaker performance. Alternatively, investors might include stocks not represented in the benchmark if they believe these selections will enhance returns. Active strategies can generate extraordinary returns or, conversely, underwhelming ones, and the Brinson-Fachler model helps us to understand the sources of these outcomes (Brinson and Fachler, 1985).

1.1 Derivation of the Effects

The Brinson-Fachler model is based on sector-level comparison, not on individual asset comparison. This is one of the model's key assumptions: both portfolios (the active strategy portfolio and the benchmark) must be constructed with different assets. Otherwise, the model may yield atypical or uninformative results. Since the goal is to understand the source of superior returns, we define these as the difference between the returns of the two portfolios:

$$\mu_P - \mu_B = \sum_{i=1}^{n} \omega_{P,i}\mu_{P,i} - \sum_{i=1}^{n} \omega_{M,i}\mu_{M,i} \tag{1}$$

Here, the subscript i refers to sectors (or markets, as stated in the original paper), not to individual stocks like AAPL or MSFT. We assume that the two portfolios may not contain the same number of individual stocks, but they do include the same number of sectors (n). This assumption allows us to combine both summations:

$$\mu_P - \mu_B = \sum_{i=1}^{n} (\omega_{P,i}\mu_{P,i} - \omega_{M,i}\mu_{M,i}) \tag{2}$$

Now imagine a hypothetical portfolio using the same sectors in our portfolio but with the same weights of the benchmark. Add and subtract inside the sum:

$$\mu_P - \mu_B = \sum_{i=1}^{n} (\omega_{P,i}\mu_{P,i} - \omega_{M,i}\mu_{M,i} + \omega_{M,i}\mu_{P,i} - \omega_{M,i}\mu_{P,i}) \tag{3}$$

Now we can group common factors and separate sums again, to create two different effects:

$$\mu_P - \mu_B = \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}]\,\mu_{P,i} + \sum_{i=1}^{n} \omega_{M,i}\,[\mu_{P,i} - \mu_{M,i}] \tag{4}$$

Here we can see a primitive form of an Allocation Effect (difference between weights) and a Selection Effect (difference between assets' returns). Now we can substitute $\mu_{P,i} = \mu_{M,i} + (\mu_{P,i} - \mu_{M,i})$ in the first term of the equation:

$$\mu_P - \mu_B = \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}]\,[\mu_{M,i} + (\mu_{P,i} - \mu_{M,i})] + \sum_{i=1}^{n} \omega_{M,i}\,[\mu_{P,i} - \mu_{M,i}] \tag{5}$$

Finally, we distribute the difference of weights in the second parentheses:

$$\mu_P - \mu_B = \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}] \, \mu_{M,i} + \sum_{i=1}^{n} \omega_{M,i} \, [\mu_{P,i} - \mu_{M,i}] + \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}] \, [\mu_{P,i} - \mu_{M,i}] \tag{6}$$

Them from this equation we can separate and define the three fundamental effects Brinson and Fachler use to understand portfolios performance, the Allocation, Selection and Interaction effects:

$$\text{Allocation} = \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}] \, \mu_{M,i} \tag{7}$$

$$\text{Selection} = \sum_{i=1}^{n} \omega_{M,i} \, [\mu_{P,i} - \mu_{M,i}] \tag{8}$$

$$\text{Interaction} = \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}] \, [\mu_{P,i} - \mu_{M,i}] \tag{9}$$

However, if you read the original paper by Brinson and Fachler, you can note that the Allocation Effect is calculated using the excess returns of each benchmark's asset with respect to the benchmark as a whole $(\mu_{M,i} - \mu_M)$. This is due to an underlying assumption about both the active portfolio and the benchmark: the sum of their weights must equal one. If this holds, then the sum of the differences in weights must be zero. Thanks to this, we can add a new term to the Allocation Effect:

$$\text{Allocation} = \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}] \, \mu_{M,i} - \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}] \, \mu_M \tag{10}$$

Since the return of the benchmark (μ_M) is a constant, it can be taken out of the summation, and the result remains zero. We can now regroup the terms within the summation to obtain the Brinson-Fachler Allocation Effect:

$$\text{Allocation} = \sum_{i=1}^{n} [\omega_{P,i} - \omega_{M,i}] \, [\mu_{M,i} - \mu_M] \tag{11}$$

If the assumption about the weights holds, both Allocation Effects will be equal at the aggregate level (i.e., their sums will be the same), but they will differ at the individual level. Brinson and Fachler introduce this excess term to better understand the nature of the Allocation Effect, which covers a large section of the original Performance Attribution paper.

1.2 Understanding the Allocation Effect

The Allocation Effect measures the impact of overweighting or underweighting specific assets in a portfolio relative to the benchmark used for comparison. Here, Brinson and Fachler explain that a positive Allocation Effect arises when we overweight high-performing assets (i.e., those with returns above the benchmark) or underweight poorly performing assets. Conversely, a negative Allocation Effect occurs when we overweight underperforming assets or underweight strong performers. Table 1 can help us to better understand this idea.

Naturally, no Allocation Effect will be observed if we replicate the benchmark's allocation by weighting our assets in the same way. However, this can also occur in the rare case where the selected assets deliver the same return as the benchmark. In conclusion, building on the previous idea, at the aggregate level the Allocation Effect does not account for the benchmark returns as in Equation 7. However, it remains important at the individual level, as it helps us understand which assets to overweight and which to underweight, as in Equation 11.

Table 1: **Allocation Effect:** Impact of Relative Weights and Asset Returns

	Overweight ($\omega_{P,i} > \omega_{M,i}$)	**Underweight** ($\omega_{P,i} < \omega_{M,i}$)
Positive Prime ($\mu_{M,i} > \mu_M$)	(+) Allocation > 0	(–) Allocation < 0
Negative Prime ($\mu_{M,i} < \mu_M$)	(–) Allocation < 0	(+) Allocation > 0

1.3 Understanding the Selection Effect

In this case, it's important to remember that the Brinson-Fachler model is designed to understand the sources of excess returns at the sector level, not at the individual stock level. This is because individual stocks yield the same returns regardless of the portfolio or the weights assigned to them. Naturally, sector-level returns reflect differences in composition and asset selection—this is the source of the Selection Effect: which assets we choose, and how those choices generate extraordinary returns. If we take Equation 8 and split it:

$$\text{Selection} = \sum_{i=1}^{n} \omega_{M,i}\mu_{P,i} - \sum_{i=1}^{n} \omega_{M,i}\mu_{M,i} \tag{12}$$

Note, then, that the Selection Effect represents the difference between two portfolios that share the same weights but contain entirely different assets. The first can be understood as a hypothetical portfolio constructed using the benchmark's allocation but substituting in the portfolio's selected assets. The second is, by definition, the benchmark itself.In this way, the difference in returns is explained solely by the selection of stocks. Nevertheless, there may be cases in which both portfolios happen to produce the same returns, even if they contain different assets.

1.4 Understanding the Interaction Effect

The most difficult effect to interpret is the Interaction Effect, which captures the combined impact of overweighting or underweighting assets and selecting better- or worse-performing stocks. For the Interaction Effect to be different from zero, both the Allocation and Selection Effects must also be nonzero. Otherwise, the Interaction Effect will be zero. If we decompose Equation 9, we will obtain the interaction among four different portfolios:

$$\text{Interaction} = \sum_{i=1}^{n} \omega_{P,i}\mu_{P,i} - \sum_{i=1}^{n} \omega_{P,i}\mu_{M,i} - \sum_{i=1}^{n} \omega_{M,i}\mu_{P,i} + \sum_{i=1}^{n} \omega_{M,i}\mu_{M,i} \tag{13}$$

The first and fourth portfolios correspond, respectively, to the actual portfolio we constructed and the benchmark used for comparison. The second and third portfolios are hypothetical constructs that mix the weights and returns of the former two. This framework explicitly assumes that the portfolio is not composed of the same assets as the benchmark; otherwise, the Selection Effect would be zero, and the Allocation Effect would account entirely for the portfolio's excess returns. In other words, no Interaction Effect would be observed, meaning that all outperformance would be attributed solely to the overweighting or underweighting of sectors.

2 Brinson-Hood-Beebower Model

In 1986, Gary P. Brinson, L. Randolph Hood, and Gilbert L. Beebower expanded the original Brinson-Fachler model, simplifying it into a less complex and more intuitive framework by constructing just four portfolios and analyzing the interactions among them. Again, this is a sector-level analysis, but in contrast to the 1985 paper, it ignores the individual performance within each sector and focuses instead on the aggregate level of the portfolios.

In Brinson et al., 1986, the authors define two different strategies that an investor can follow: the **Passive Strategy** and the **Active Strategy**. In the former, the investor (perhaps more risk-averse) chooses to simply buy the benchmark portfolio (for example, IVV, the ETF tracking the S&P 500) and generate 'naive' returns. In contrast, the latter strategy involves the investor constructing their own portfolio by selecting assets and weighting them according to their own criteria. Then, the active and the passive portfolios will be essential for the understanding of this model. From Equation 1 define these portfolios:

$$\mu_a - \mu_p = \sum_{i=1}^{n} \omega_{a,i}\mu_{a,i} - \sum_{i=1}^{n} \omega_{p,i}\mu_{p,i} \tag{14}$$

Here, we use the subscript a to refer to the active strategy and p to refer to the passive strategy. These two portfolios can be used to construct alternative portfolios that combine elements of both active and passive strategies. For example, overweighting or underweighting the same asset composition as the benchmark results in a portfolio with active allocation (or 'timing', as referred to in the paper) and passive selection. Table 2 presents the return decomposition for the four portfolios constructed from combinations of active and passive strategies. Note that Quadrant IV and Quadrant I correspond to the pure active and pure passive strategies, respectively.

Table 2: **BHB Model:** Returns Accountability

	Active Selection	**Passive Selection**
Active Allocation	(IV) $\sum_{i=1}^{n} \omega_{a,i}\mu_{a,i}$	(II) $\sum_{i=1}^{n} \omega_{a,i}\mu_{p,i}$
Passive Allocation	(III) $\sum_{i=1}^{n} \omega_{p,i}\mu_{a,i}$	(I) $\sum_{i=1}^{n} \omega_{p,i}\mu_{p,i}$

2.1 Calculating the Effects

From Table 2, we can calculate the Allocation, Selection, and Interaction Effects as defined by Brinson et al. In this framework, the authors define the passive strategy (Quadrant I) as the **Investment Policy**, which refers to the returns an investor obtains by simply accepting what the market provides. Therefore, the only way to generate higher returns is through **'Market Timing'** (allocation) and **'Security Selection'**. First, let us define the Market Timing Effect as the difference between the Active-Allocation–Passive-Selection Portfolio and the Benchmark Portfolio (Quadrant II minus Quadrant I):

$$\text{Timing} = \sum_{i=1}^{n} \omega_{a,i}\mu_{p,i} - \sum_{i=1}^{n} \omega_{p,i}\mu_{p,i} \tag{15}$$

It is easy to demonstrate (and it can be a good exercise for the reader) that Equation 7 and Equation 15 represent the same effect. What we are trying to capture is how much of the excess return can be attributed to the variation in portfolio weights. This method of calculating the effects is quite intuitive, as it involves simply taking the difference in returns between two portfolios that share the benchmark's security selection but apply different allocation strategies. Same logic can be used to obtain the Security Selection Effect:

$$\text{Selection} = \sum_{i=1}^{n} \omega_{p,i}\mu_{a,i} - \sum_{i=1}^{n} \omega_{p,i}\mu_{p,i} \tag{16}$$

In this case, we are taking the difference between Quadrant III and Quadrant I, which means we are subtracting the Benchmark from the Passive-Allocation–Active-Selection portfolio. The underlying logic is that, by maintaining a passive allocation policy, any differences in returns must be attributed to variations in selection strategies. It is not difficult to demonstrate that Equation 8 and Equation 16 represent the same effect. Finally, the calculation of the Interaction Effect will imply the use of all the portfolios:

$$\text{Interaction} = \sum_{i=1}^{n} \omega_{a,i}\mu_{a,i} + \sum_{i=1}^{n} \omega_{p,i}\mu_{p,i} - \sum_{i=1}^{n} \omega_{a,i}\mu_{p,i} - \sum_{i=1}^{n} \omega_{p,i}\mu_{a,i} \tag{17}$$

For the Interaction Effect, we compute the difference between the diagonals: adding Quadrants I and IV, and then subtracting the sum of Quadrants II and III. As in the Brinson-Fachler model, this Interaction Effect captures the same concept (note that Equation 13 and Equation 17 represent the same effect). By comparing the 'mixed-strategy' portfolios to the 'pure-strategy' portfolios, we isolate the effect of simultaneously varying both asset weights and security selection in the portfolio.

$$\begin{aligned}
\text{Timing} &= Q_{II} - Q_I \\
\text{Selection} &= Q_{III} - Q_I \\
\text{Interaction} &= Q_{IV} - Q_{III} - Q_{II} + Q_I \\
\hline
\text{Total} &= Q_{IV} - Q_I
\end{aligned}$$

Unsurprisingly, the sum of the three effects equals the excess returns generated by our pure active-strategy portfolio (Quadrant IV minus Quadrant I). The BHB model is essentially an alternative way to interpret the Brinson-Fachler model; both explain the same phenomena using the same decomposition. The only difference is that the former operates at the aggregate portfolio level, while the latter focuses on the sector level.

3 Menchero-Davis Risk Attribution

After examining the sources of portfolio returns, we now turn to the origins of its risk. To do this, we adopt a methodology proposed by José Menchero and Ben Davis known as the **x-sigma-rho** decomposition. Although mathematically rigorous, the intuition behind this approach is simple and insightful, as we shall see (Menchero and Davis, 2011).

3.1 X-Sigma-Rho Decomposition

The Menchero-Davis methodology begins with the traditional definition of portfolio variance, as introduced in Module 2. Here, we understand variance as a weighted average of the covariances among all the stocks in the portfolio. For a portfolio with M stocks, the variance can be defined as:

$$\sigma_P^2 = \sum_{m=1}^{M} \sum_{n=1}^{M} x_m x_n \text{Cov}(\mu_m, \mu_n) \tag{18}$$

Here, x_m denotes the exposure to asset m—that is, the fraction of total capital invested in that asset. Unlike traditional formulations that use ω_m for weights, this notation highlights the role of exposures in driving risk. We can rearrange the summation to isolate the contributions of each asset:

$$\sigma_P^2 = \sum_{m=1}^{M} x_m \left[\sum_{n=1}^{M} x_n \text{Cov}(\mu_m, \mu_n) \right] \tag{19}$$

This expression makes it clear that the portfolio's risk results from interactions between each asset and all others via their covariances. To develop further intuition, we can expand the inner summation explicitly:

$$\sigma_P^2 = \sum_{m=1}^{M} x_m \left[x_1 \mathrm{Cov}(\mu_m, \mu_1) + x_2 \mathrm{Cov}(\mu_m, \mu_2) + ... + x_M \mathrm{Cov}(\mu_m, \mu_M) \right] \tag{20}$$

Since all covariance terms involve R_m, we can exploit the linearity of covariance to write:

$$\sigma_P^2 = \sum_{m=1}^{M} x_m \mathrm{Cov}(\mu_m, x_1\mu_1 + x_2\mu_2 + ... + x_M\mu_M) \tag{21}$$

We now join the second term of the covariance in a summation. Then, recalling the definition of portfolio return, we arrive at an alternative expression to understand portfolio variance.

$$\sigma_P^2 = \sum_{m=1}^{M} x_m \mathrm{Cov}(\mu_m, \sum_{n=1}^{M} x_n\mu_n) = \sum_{m=1}^{M} x_m \mathrm{Cov}(\mu_m, \mu_P) \tag{22}$$

Now we find an interesting result: the variance of a portfolio depends on the covariance between each asset and the portfolio itself. This may seem counterintuitive, but it is a useful result that allows us to derive the **x-sigma-rho** decomposition. Recalling the definition of the correlation coefficient (check Module 1), we can substitute the covariance term in Equation 23:

$$\sigma_P^2 = \sum_{m=1}^{M} x_m \sigma_P \sigma_m \rho_{m,P} \tag{23}$$

Since the portfolio's volatility is a constant, we can factor it out of the summation. Then, by dividing both sides of the equation by the portfolio's volatility (σ_P), we arrive at the final decomposition:

$$\sigma_P = \sum_{m=1}^{M} x_m \sigma_m \rho_{m,P} \tag{24}$$

Equation (24) shows that portfolio volatility can be expressed as a linear combination of each asset's volatility, weighted by its exposure and correlation with the portfolio. This representation, known as the **x-sigma-rho decomposition**, allows us to understand how each asset contributes to total risk—not just through its own volatility, but also through how it co-moves with the rest of the portfolio.

3.2 Marginal Contribution to Risk

After deriving the **x-sigma-rho** decomposition, we can interpret the portfolio's volatility as a linear function in M dimensions (essentially a hyperplane) dependent on the volatilities of the individual assets (σ_m). The additive property of the portfolio's risk is mathematically convenient and we can derive interesting conclusions from it. We can separate the summation and rename the interaction term involving sigma and rho as alpha:

$$\sigma_P = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \cdots + \alpha_M\sigma_M \tag{25}$$

Then, we can interpret each linear term as the **Contribution to Risk**, if we divide Equation 25 by the portfolio's volatility, we obtain the **Percentage Contribution to Risk**, which—naturally—must sum to 1 (or 100%). Another important concept to understand is: how much riskier will our portfolio become if we increase exposure to a certain asset? The **Marginal Contribution to Risk** of each asset is calculated by taking the derivative of Equation 26 with respect to the exposure to asset m:

$$\frac{\partial \sigma_P}{\partial x_m} = \mathrm{MCR_m} = \sigma_m \rho_{m,P} \tag{26}$$

Thanks to the derivation of the MCR, we can express the portfolio's volatility as the weighted average of the Marginal Contributions to Risk (MCR) of each asset. This helps us draw an analogy between volatility and returns, since portfolio returns are also computed as the weighted average of individual asset returns:

$$\sigma_P = \sum_{m=1}^{M} x_m \mathrm{MCR}_m \tag{27}$$

This result mirrors the structure of portfolio returns, which are also a weighted average of individual asset returns. The analogy reinforces a key insight: just as returns can be attributed to assets, so can risk.

3.3 Correlation Drilldown

The Menchero methodology also proposes an alternative way to understand the correlation between an individual stock and the portfolio. The **Correlation Drilldown** helps us identify the sources of correlation between an asset and the portfolio to which it belongs. Recalling the definition of the correlation:

$$\rho_{m,P} = \frac{\mathrm{Cov}(\mu_m, \mu_P)}{\sigma_m \sigma_P} \tag{28}$$

Substituting the definition of the portfolio's returns into the previous equation:

$$\rho_{m,P} = \frac{1}{\sigma_m \sigma_P} \mathrm{Cov}(\mu_m, \sum_{n}^{M} x_n \mu_n) \tag{29}$$

Splitting the sum and applying the linearity of the covariance:

$$\rho_{m,P} = \frac{1}{\sigma_m \sigma_P} \sum_{n=1}^{M} x_n \mathrm{Cov}(\mu_m, \mu_n) \tag{30}$$

Replace the covariance by its definition using the standard deviations and the correlation coefficient:

$$\rho_{m,P} = \frac{1}{\sigma_m \sigma_P} \sum_{n=1}^{M} x_n \sigma_m \sigma_n \rho_{m,n} \tag{31}$$

The volatility of stock m can be canceled out, since it appears in both the numerator and the denominator. After rearranging the equation, we arrive at the decomposition of the correlation:

$$\rho_{m,P} = \sum_{n=1}^{M} x_n \left(\frac{\sigma_n}{\sigma_P} \right) \rho_{m,n} \tag{32}$$

What we can conclude from the previous equation is that the correlation of stock m with the portfolio is a function of its correlations with the other stocks in the portfolio. If we imagine an equally weighted portfolio composed of only two assets, and if they are negatively correlated, then the correlation of each stock with the portfolio will naturally be negative as well. While this may seem like an obvious result, it helps us understand why certain stocks that are theoretically negatively correlated may end up being positively correlated—simply because both are positively correlated with the market.

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